Block Diagram Manipulations

The Basic Control Loop

In this section we will examine methods for simplifying systems of transfer functions to a single function. As the first example, we will consider a generic control loop. In this system, \( G_c \) represents a control algorithm. \( G_a \) is the transfer function for the actuator. The function of an actuator is to take the output of the controller (usually a low power electrical signal) and convert it to something that can alter the process; for example, a heater or an agitator. \( G_p \) is the transfer function for the process. \( H \) is the transfer function for the sensor (e.g., a thermometer).

This system has two inputs and one output. \( R \) is the setpoint (or desired output) for the system. When the output (via the sensor) is subtracted from the setpoint, the resulting error, \( \varepsilon \), is used as the input for the controller. The second input, \( U \), is the load or disturbance input. This input is usually not under the engineer’s control. Some examples of disturbances might be feedstock entering a reactor or external temperature changes. The output variable, \( C \), is the variable being controlled.

In general, there are two categories of control problems. In “Servo” problems, there is no change in load, but the setpoint changes. In “Regulator” problems, the setpoint remains constant, but the load changes. In real systems both the load and the setpoint may change simultaneously.

\[
\begin{align*}
R &= \text{Setpoint} \\
\varepsilon &= \text{Error} \\
U &= \text{Load / Disturbance Variable} \\
B &= \text{Variable Produced by Measuring Element} \\
C &= \text{Controlled Variable} \\
M &= \text{Manipulated Variable}
\end{align*}
\]
1. Determining transfer function for relating $R$ to $C$, the servo problem.

   A. These are “overall” transfer functions (i.e., include entire process), so we are looking for \( \frac{C}{R} \).

   B. Remember, these are differences from steady state, not absolute values. If the only change is in $R$, $U = 0$.

Then, by combining $G_C$, $G_A$, and $G_P$ into a single transfer function “$G_1$” ($G_1 = G_C \cdot G_A \cdot G_P$).
This loop can be solved using three simultaneous equations:

\[ \begin{align*}
C & = G_1 \varepsilon \\
B & = HC \\
\varepsilon & = R - B
\end{align*} \]

III into I yields

\[ \begin{align*}
C & = G_1 (R - B)
\end{align*} \]

Substitute in II

\[ \begin{align*}
C & = G_1 (R - HC) \\
C & = G_1 R - HG_1 C \\
C + HGC & = G_1 R \\
C & = \frac{G_1 R}{1 + G_1 H} \\
\frac{C}{R} & = \frac{G_1}{1 + G_1 H} \\
& = \frac{G_1 G_A G_p}{1 + G_c G_A G_p H}
\end{align*} \]

2. Determining the transfer function relating \( U \) to \( C \), the regulator problem.

A. As before, we are looking for the overall transfer function relating \( C \) to \( U \). So the goal is \( \frac{C}{U} \).

B. In this case, \( U \) is changing, not \( R \). Therefore, \( R \) is 0 in deviation terms.
\[ G_2 = G_C \quad G_A \]

\[ C = (U + M)G_p \]

\[ M = G_2 \varepsilon \]

\[ \varepsilon = -B \]

\[ B = HC \]

Then

\[ C = (U + G_2 \varepsilon)G_p \]

\[ C = (U + G_2 (-B))G_p \]

\[ C = (U + G_2 (-HC))G_p \]
\[ C = U G_p - G_2 G_p HC \]

\[ C + G_2 G_p HC = U G_p \]

\[ C \left( 1 + G_2 G_p HC \right) = U G_p \]

\[
\begin{align*}
C &= U \frac{G_p}{1 + G_2 G_p H} \\
C &= U \frac{G_p}{1 + G_c G_A G_p H} \\
C &= \frac{G_p}{U} \frac{1}{1 + G_c G_A G_p H}
\end{align*}
\]

If changes occur in both \( R \) and \( U \), then we can solve the system by adding the two solutions together.

So, if

\[
\frac{C}{R} = \frac{G_c G_A G_p}{1 + G_c G_A G_p H}
\]

And

\[
\frac{C}{U} = \frac{G_p}{1 + G_c G_A G_p H}
\]

The response of the system to both changes is

\[
C = R \left( \frac{G_c G_A G_p}{1 + G_c G_A G_p H} \right) + U \left( \frac{G_p}{1 + G_c G_A G_p H} \right)
\]

Or

\[
C = \frac{R G_c G_A G_p + U G_p}{1 + G_c G_A G_p H}
\]
Start by eliminating the innermost loop.

This has the transfer function

\[
\frac{Y}{X} = \frac{G_{c2}G_A}{1 + G_{c2}G_A H_2} = G\alpha
\]

Note: What if this was a positive feedback loop; i.e.,

Then transfer function

\[
\frac{Y}{X} = \frac{G_{c2}G_A}{1 - G_{c2}G_A H_2}
\]
Because we are solving the servo problem, $U_1$ and $U_2$ are 0. Now we have:

![Diagram of control system]

This becomes

![Diagram of simplified control system]

Where

$$G_\beta = G_A G_\alpha G_{P1} G_{P2}$$

$$\frac{C}{R} = \frac{G\beta}{1 + G\beta H_1}$$

So

$$\frac{C}{R} = \frac{G_{C1} G_A G_{P1} G_{P2}}{1 + G_{C1} G_A G_{P1} G_{P2} H_1}$$
Some Additional Block Diagram Manipulation Techniques

In some cases, simplification of the block diagram will require altering the order of the various elements of a block diagram. Some of the possible operations are described below.

1. Moving a summing junction behind a block

\[ X_3 = G(X_1 \pm X_2) \]

2. Moving a summing junction ahead of a block

\[ X_3 = GX_1 \pm GX_2 \]

\[ X_3 = GX_1 \pm X_2 \]
3. Altering the order of summing junctions

\[
X_3 = (X_1 + (1/G)X_2)G = GX_1 + 1/G \cdot X_2 = GX_1 + X_2
\]

\[
X_4 = (X_1 - X_2) + X_3 = X_1 - X_2 + X_3
\]

Is equivalent to

\[
X_4 = (X_1 + X_3) - X_2 = X_1 - X_2 + X_3
\]
4. Moving a pick off point ahead of a block

\[ X_{2A} = G \cdot X_1 \]
\[ X_{2B} = G \cdot X_1 \]

5. Moving a pick off point behind a block

\[ X_2 = G \cdot X_{1A} \]
\[ X_2 = G \cdot X_{1B} \]
\[ X_{1B} = \frac{X_2}{G} \]
6. Parallel forward paths

Becomes

\[ G_1 + G_2 \]
A Complex System with Interlaced Control Loops

Here is another complex example to practice simplifying block diagrams. The challenging issue is the interlacing of the feedback loops. They need to be converted to nested loops.

In this case, we will manipulate the feedback loop containing $H_2$ to be outside the loop containing $H_1$. Then moving the summing junction before $G_1$, the highlighted area (indicated by dotted line) may be written:
We can now swap the order of the summing junctions to yield

\[
\frac{G_1 G_2}{1 - G_1 G_2 H_1}
\]

Note the sign change (since this is positive feedback loop)
Let

\[ \frac{A}{B} = \frac{G_1G_2G_3}{1 - G_1G_2H_1} \]

Then the transfer function for the highlighted area is

\[ \frac{A}{B} \cdot \frac{1}{1 + \frac{AH_2}{BG_1}} \]

\[ \frac{A}{B} \cdot \frac{1}{BG_1 + AH_2} \]

\[ \frac{A}{B} \cdot \frac{BG_1}{BG_1 + AH_2} \]

\[ \frac{AG_1}{BG_1 + AH_2} = \frac{G_1G_2G_3}{G_1(1 - G_1G_2H_1) + G_1G_2G_3H_2} \]

\[ = \frac{G_1G_2G_3}{1 - G_1G_2H_1 + G_2G_3H_2} = \frac{D}{E} \]

And the diagram becomes

\[ \text{Note } H_3 = 1 \]
Then the transfer function becomes

\[ \frac{D}{E} \left(1 + \frac{D}{E}\right) \]

So

\[ \frac{D}{E} \]

\[ \frac{D}{E + D} \]

So

\[ \frac{D}{E} \cdot \frac{E}{D + E} \]

And

\[ \frac{D}{D + E} \]

Substituting back in for \(D\) and \(E\).

\[ \frac{G_1G_2G_3}{1 - G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3} \]

Or

\[ \frac{G_1G_2G_3}{1 - G_1G_2H_1 + G_2G_3H_2 + G_1G_2G_3} \]